Homoclinic Bifurcation and Endogenous Cycles in the Dynamic IS-LM Model

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Abstract: This paper contributes to the new keynesian literature by showing that stable endogenous cycles can emerge as equilibrium solutions of the traditional IS-LM model. The application of the original Bogdanov-Takens theorem allows us to determine the regions of the parametric space where the model exhibits a global indeterminate solution, and a low-growth trapping region, characterized by a continuum of equilibrium trajectories in the proximity of a homoclinic bifurcation.

Keywords: Multiple steady states, homoclinic bifurcation, oscillating solutions.

1. INTRODUCTION

Recent literature has revived attention on the nonlinear dynamic properties of the celebrated IS-LM model to explain the persistence of endogenous fluctuations in a macro-economic system, and justify its theoretical relevance for both fiscal and monetary stabilization policies to act as a possible selection device among different equilibria (Makovinyiova and Zimka 2009; Kaličinská 2012).

In general, the interest for the traditional IS-LM framework is largely undermined by the severe functional restrictions needed to generate the required oscillating pattern. In particular, a Kaldorian *S*-shaped investment function, is generally appealed by this literature to show the emergence of cyclical solutions (Schinasi 1981, 1982; Lorenz 1993; Sasakura 1994; Bischi *et al*. 2001). Unfortunately, the adoption of non-linear functions increases the difficulties in handling this model.

To overcome this problem, analyses of phase transitions from a determinate equilibrium to stable oscillations, and potentially chaotic motion, are explained either by imposing time-delayed feedbacks in the tax collection function (Cai 2005; De Cesare and Sportelli 2005; Fanti and Manfredi, 2007; Neamtu *et al*. 2007; Tu *et al*. 2013), or by looking at some specific parameter regions through the standard Hopf bifurcation theorem (Gandolfo 1997; Makovinyiova 2011; Guirao *et al*. 2012; Neri and Venturi 2007). However, most of this literature confines herself entirely on the grounds of a local analysis (Slobodyan 2007; Chamley 1993; Benhabib and Farmer 1994, 1996; Benhabib and Perli 1994; Benhabib *et al*. 1994, 2000), and so lacks providing a complete picture of the dynamics emerging outside the small neighborhood of the steady state, to whom we refer to as global *indeterminacy*.

This paper aims to show that all the government attempts to stabilize output through fiscal policies, when aggregate demand fluctuates around a trend, might produce a destabilizing effect in the full system dynamics. To prove this, we apply the Bogdanov-Takens bifurcation theorem (henceforth, *BT*), a global analysis tool which allows us to prove that a trajectory, starting in the vicinity of a saddle steady state can approach from outside a limit cycle enclosing a non saddle steady state, and characterize the regions of the parametric space where the model gives rise to a global indeterminate equilibrium, where active policies might not be able to avoid the emergence of a low-growth trapping region.[1]

The paper develops as follows. Section 2 introduces the model, specifies the functions, and derives the long-run equilibrium. Section 3 points out the stability properties of the equilibrium, and shows the conditions for the emergence of stable limit cycles. Section 4 applies the Bogdanov-Takens theorem to prove the possibility of global indeterminacy in presence of a homoclinic bifurcation. Some examples are also discussed to validate the model. A brief conclusive section reassesses the main findings. All necessary proofs are provided in Appendix.

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[1]The Bogdanov-Takens bifurcation is a powerful mathematical tool in simplifying highly non-linear dynamical system, and is largely used in mathematics, physics and biology, but has found a surprisingly limited application in economics (Benhabib et al. 2001; Bella and Mattana 2014).

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2. THE MODEL

Assume the simple dynamic fixed-price Schinasi's variant of the IS-LM model for an open economy, with pure money financing of the budget deficit which, in the case of an instantaneous adjustment in the money market, implies the following system of first order differential equations (Schinasi 1981, 1982; Makovinyiova 2011)

$$\begin{aligned} \dot{R} &= \delta\left[L(R,Y) - M\right] \\ \dot{Y} &= \alpha\left[I(Y,R) - S(Y^D,R) + G - T(Y) + N(Y)\right] \quad (P) \\ \dot{M} &= G - T(Y) + N(Y) \end{aligned}$$

where dots stand for time-derivatives. It is assumed that all functions are continuously differentiable at a suitable order. $L(R,Y)$ is the liquidity function, which relates the demand for money to the (real) interest rate, *R*, and the income level, *Y*. $I(R,Y)$ is the investment function, which is assumed to depend on income and on the interest rate. $S(R,Y^D)$ represents savings as a function of both disposable income and the interest rate as a further argument (Cai 2005; Makovinyiova 2011). *M* describes the nominal money supply. *T(Y)* is the tax collection function, which only depends on income. Finally $G > 0$ is the (constant) government expenditure, whereas α and δ are scale parameters. In addition, following standard textbooks, we introduce international trade by assuming a net export function, $N(Y) = \bar{N} - qY$, where $\bar{N}$ is a fixed amount, and *q* is the marginal propensity to import out of income. Since we consider fixed exchange rates, the trade-balance disequilibrium leads to an immediate change in both the money stock and the excess demand for goods.

For the sake of a simple representation, we shall assume a tax function linear in *Y*, so that $T(Y) = \tau Y$. Hence, $Y^D = (1-\tau)Y$.

Whereas there is no theoretical and empirical disagreement in the literature on the following derivatives

$$L_Y > 0; L_R < 0$$

And

$$I_R < 0; I_Y > 0; S_Y > 0$$

less clear is the sign of S_R, which remains ambiguous (Abrar 1989).[2] Additionally, since $I_{YY} \neq 0$, the gross investment is expected to behave in a sigmoid Kaldorian fashion.[3] We show that this is crucial for the scopes of the paper.

2.1. Steady State

Let (R^*, Y^*, M^*) be values of (R, Y, M) in (*P*) such that $\dot{R} = \dot{Y} = \dot{M} = 0$. Simple algebra shows that, at the steady-state, we have

$$H^*(R,Y) = 0 \qquad (1.1)$$

$$M^* = L^*(R,Y) \qquad (1.2)$$

$$Y^* = \frac{G + \bar{N}}{\tau + q} \qquad (1.3)$$

where we group total inventories as

$$H^*(R,Y) = I^*(R,Y) - S^*(R,(1-\tau)Y) \qquad (2)$$

to simplify notation.

Conditions for existence and uniqueness of the steady state follow consequently. Let

$$\phi = H(R,Y) \qquad (3)$$

with ϕ conveniently smooth in all its arguments. Let also ϕ_R and ϕ_{RR} be the first and second-order derivatives of ϕ with respect to *R*. If Alternatively, if ϕ_R changes sign

Let now $\omega \in \Omega$ is a point of the parameter space. Then

Lemma 1 Let $\hat{\Omega} \equiv \{\omega \in \Omega : \phi_R > 0 \text{ or } \phi_R < 0\}$. Then, if $\omega \in \hat{\Omega}$, ϕ_R has a definite sign in the domain

[2]It is commonly thought that savings would increase when interest rate rises. However, the relationship between interest rate and savings is more complex and uncertain. Economists explain this ambiguity by distinguishing a substitution effect from an income effect, so that, depending on which effect prevails, the economic agent is a net lender or a net borrower (Blanchard and Fisher 1989).
Empirical evidence for Italian data seems to support this findings. In details, $S_R > 0$ in the 70's and during the period 2000-2005; whereas $S_R < 0$ for the whole 80's and 90's (Baffigi 2011).

[3]As output increases above its natural level, agents will invest at a decreasing rate, since a government intervention to stabilize the economy is expected when output moves further away from his trend. This renders the intermediate-run equilibrium locally unstable. To avoid this, the government might set the level of public spending or the amount of taxes to stabilize the economy, but this presumes that the policy-maker knows at each point in time where the economy is along the cycle, which is unrealistic.

$D=\{(R):\ R>0\}$, the function ϕ monotonically decreases with the interest rate, and only one intersection (i.e., *one steady state*) with the R-axis occurs. Consider now $\omega \in \breve{\Omega}$, where, $\breve{\Omega} \equiv \{\omega \in \Omega : \phi_R$ changes sign at $R=\hat{R}\}$. Assume $\phi_{RR}>0$. Then, if (i) $\phi(\hat{R})<0$, there are two steady states, one with a low interest rate $\left(R_-^*, Y^*\right)$ and one with a high interest rate $\left(R_+^*, Y^*\right)$; (ii) $\phi(\hat{R})=0$, there is one steady state; (iii) $\phi(\hat{R})>0$, there are no steady states (see, Figure **1**). The reverse statements apply for $\phi_{RR}<0$.

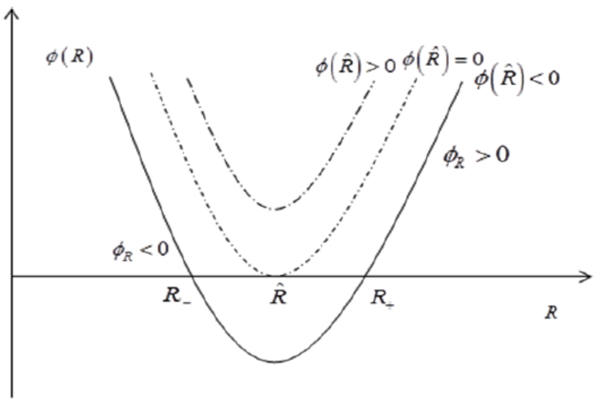

Figure 1: The $\phi(R)$ function.

Proof Let $\omega \in \hat{\Omega}$. Since, by assumption, its first derivative does not vanish in *D*, the function $\phi(R)$ is always monotonically decreasing/increasing in *D* and only one steady state is possible. Conversely, if $\omega \in \breve{\Omega}$, $\phi(R)$ follows a parabolic evolution, and multiple intersections (i.e., multiple equilibria) with the *R*-axis may occur.

3. LOCAL STABILITY ANALYSIS

Consider the Jacobian matrix associated to (*P*), at the steady state

$$\mathbf{J}^* = \begin{bmatrix} \delta L_R^* & \delta L_Y^* & -\delta \\ \alpha H_R^* & \alpha\left(H_Y^* - \tau - q\right) & 0 \\ 0 & -\tau - q & 0 \end{bmatrix} \tag{4}$$

where, for the sake of a simple notation, the arguments of the partial derivatives have been dropped. Consider the characteristic polynomial

$$\mathrm{Det}\left(\lambda\mathbf{I}-\mathbf{J}^*\right) = -\lambda^3 + \mathrm{Tr}(\mathbf{J}^*)\lambda^2 - \mathrm{B}(\mathbf{J}^*)\lambda + \mathrm{Det}(\mathbf{J}^*) \tag{5}$$

where **I** is the identity matrix, and

$$\mathrm{Tr}(\mathbf{J}^*) = \alpha\left(H_Y^* - \tau - q\right) + \delta L_R^*$$
$$\mathrm{Det}(\mathbf{J}^*) = \alpha\delta(\tau+q)H_R^*$$
$$\mathrm{B}(\mathbf{J}^*) = \alpha\delta\left[L_R^*\left(H_Y^* - \tau - q\right) - H_R^* L_Y^*\right]$$

are the trace, the second order sum of principal minors, and the determinant of $\mathbf{J}^*$, respectively.

To study the stability properties in a planar system from the local analysis perspective, it is crucial to establish the signs of both $\mathrm{Det}(\mathbf{J}^*)$ and $\mathrm{Tr}(\mathbf{J}^*)$. The neat Routh-Hurwitz criterion applies, showing that necessary conditions for the emergence of attracting orbits imply that

$$H_R^* < 0 \tag{6.1}$$

$$\alpha\left(H_Y^* - \tau - q\right) + \delta L_R^* > 0 \tag{6.2}$$

which guarantee that the steady state is an unstable node or focus. More precisely

Proposition 1 Recall Lemma 1. Let $\omega \in \hat{\Omega}$ and first assume $H_R^*>0$. Then, the (unique) steady state is an unstable node or focus if (6.2) is satisfied. Conversely, if $H_R^*<0$, the steady state is a saddle.

Let now $\omega \in \breve{\Omega}$ and assume first $\phi_{RR}>0$. As shown in Lemma 1, we can either have a dual steady state, one steady state or no steady states at all. In the former case, at $\left(R_-^*, Y^*\right)$, $H_R^*>0$, so that the low interest rate equilibrium is an unstable node or focus if (6.2) is satisfied. The other steady state $\left(R_+^*, Y^*\right)$ has $H_R^*<0$ and is therefore a saddle. The low interest rate steady state and the high interest rate steady state interchange their stability properties if $\phi_{RR}<0$.

Proof To exclude a saddle we need $\mathrm{Det}(\mathbf{J}^*)>0$, which happens if $H_R^*>0$ applies. Furthermore if (6.2) is satisfied, $\mathrm{Tr}(\mathbf{J}^*)>0$, and the steady state is an unstable node or focus.

As well discussed in the literature, we obtain that

Remark 1 This characteristic can be justified in a Kaldorian perspective, namely with the assumption of a S-shaped investment function (Schinasi 1981).

which implies also that

Corollary 1 *From the perspective of the local analysis, only a Kaldorian-type economy, satisfying conditions in* (6.*i*) *can give rise to stable limit cycles.*

Proof In the neighborhood of the non-saddle steady state we can have oscillating solutions if $H_R^* > 0$. In this case, since $L_R^* < 0$, $\mathrm{Tr}(\mathbf{J}^*)$ has positive sign only if $\alpha\left(H_Y^* - \tau - q\right) + \delta L_R^* > 0$, which implies a greater-than-unity marginal propensity to spend out of income.

In the next section, we use specific functional forms, joint with some numerical examples, to characterize the regions of the parametric space where the model exhibits a global indeterminate solution, and a low-growth trapping region, for an economically plausible range. In details, we assume $I = \beta \frac{Y^\eta}{R^\gamma}$ as in Neamtu *et al*. (2007), we set $L = kY - hR$ as in Makovinyiova (2011), while we maintain a general savings function.

4. THE BOGDANOV-TAKENS BIFURCATION

In this section, we discuss the application of the *BT* theorem to system (*P*). The theorem allows us to detect a particular type of global phenomenon, namely the homoclinic bifurcation, by which orbits growing around the non-saddle steady state collide with the saddle one. As shown hereafter, this phenomenon can be used to establish the possibility of global indeterminacy of the equilibrium. Hence, let us give the following

Definition 1 Given the fixed point $\left(R^*, Y^*, M^*\right)$, and the associated Jacobian matrix $\mathbf{J}^*$, system (P) undergoes a Bogdanov-Takens bifurcation if the linearization of $\mathbf{J}^*$ around that point has a double-zero eigenvalue.

Basically, assuming that some non-degeneracy conditions are satisfied, the *BT* singularity is referred to as a co-dimension two bifurcation, that is to say two parameters must be varied for such bifurcation to occur. Therefore,

Lemma 2 Let $(\overline{\tau}, \overline{\beta})$ be the values for which simultaneously $\mathrm{B}(\mathbf{J}^*) = 0$ and $\mathrm{Det}(\mathbf{J}^*) = 0$. Specifically, $\overline{\tau} = \frac{\beta\eta\left(\frac{G+\bar{N}}{\tau+q}\right)^{\eta-1}}{(1-S_Y)R^\gamma} - \frac{q+S_Y}{(1-S_Y)}$ and $\overline{\beta} = R^\gamma S\left(\frac{G+\bar{N}}{\tau+q}\right)^{-\eta}$. Then, for $\tau = \overline{\tau}$ and $\beta = \overline{\beta}$, the linearization matrix $\mathbf{J}^*$ has a zero eigenvalue of multiplicity two, and a third eigenvalue given by $\mathrm{Tr}(\mathbf{J}^*)$.

Proof As shown in Appendix, the candidate bifurcation values $(\overline{\tau}, \overline{\beta})$ are obtained by equating to zero, and thus combining, $\mathrm{B}(\mathbf{J}^*)$ and $\mathrm{Det}(\mathbf{J}^*)$, given steady state values in (*P*).

Next we use a two-dimensional center manifold reduction to put system (*P*) in a truncated Jordan canonical form, and introduce the following auxiliary variables $\mu = \beta - \overline{\beta}$ and $\nu = \tau - \overline{\tau}$. The systematic procedure given by Wiggins (1991) allows us to obtain the following

Lemma 3 For parameter values (τ, β) sufficiently close to $(\overline{\tau}, \overline{\beta})$ the vector field in (P) is topologically equivalent to the following system (joint with $\dot{\mu} = 0$ and $\dot{\nu} = 0$)

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \mathbf{A}(\mu, \nu) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} \overline{F}_1 \\ \overline{F}_2 \end{pmatrix} \tag{7}$$

where $\mathbf{A}(\mu, \nu)$ is a matrix that vanishes at $(\mu, \nu) = (0,0)$, while $\overline{F}_1$ and $\overline{F}_2$ contain the high order nonlinear terms.

Proof See Appendix.

The system (7) can be further simplified *via* normal form theory and time rescaling. A transverse family of this vector field (i.e., versal deformation) can be lastly found, such that the study of the local dynamics can be used to infer the presence of the global bifurcation in the original vector field, (*P*). For our economy, we can show the following

Proposition 2 The transverse family

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= \varepsilon_1 w_1 + \varepsilon_2 w_2 + w_1^2 + s w_1 w_2 \qquad s = \pm 1 \end{aligned} \tag{8}$$

is a topologically equivalent versal deformation of (7). The unfolding parameters, ε_1 and ε_2, are functions of $\mu = \beta - \overline{\beta}$ and $\nu = \tau - \overline{\tau}$, and satisfy the transversality condition $\frac{\partial(\varepsilon_1, \varepsilon_2)}{\partial(\mu, \nu)}\big|_{\mu=\nu=0} \neq 0$. Therefore, the change of the original bifurcation parameters (τ, β) through ε_1 and ε_2 is a local diffeomorphism.

Proof See Appendix for all necessary computations.

The global bifurcation associated with system (8) crucially depends on the sign of *s*. If we assume $s = +1$,[4] it follows that

Lemma 4 Recall Proposition 1. For parameter values (τ, β) sufficiently close to $(\overline{\tau}, \overline{\beta})$ we can identify:

[4]The case where $s = -1$ is very similar, so we leave it out of this demonstration.

i) a curve $\mathbf{N} \equiv \left\{ \left(\varepsilon_1, \varepsilon_2\right) : \varepsilon_1 = -\frac{49}{25}\varepsilon_2^2 + \mathrm{O}\left(\varepsilon_2^{5/2}\right),\ \varepsilon_2 > 0 \right\}$ corresponding to a saddle homoclinic bifurcation; and ii) a curve $\mathbf{M} \equiv \left\{ \left(\varepsilon_1, \varepsilon_2\right) : \varepsilon_1 = -\varepsilon_2^2,\ \varepsilon_2 > 0 \right\}$ corresponding to a (sub-critical) Andronov-Hopf bifurcation. For the scope of our analysis, we will henceforth concentrate on the case $\varepsilon_1 < 0$.

In case of a dual steady state, it is indeed true that one steady state is a saddle (S) whereas the other steady state is non saddle (NS). In particular, according to the parameters specification being considered:

i) the NS equilibrium located above the curve $\mathbf{M}$ is a source, and there exists a heteroclinic connection leading from NS to S;

ii) the NS equilibrium located below the curve $\mathbf{N}$ is a sink, there exists a heteroclinic connection leading from S to NS;

iii) the NS equilibrium located between the two curves $\mathbf{M}$ and $\mathbf{N}$ is an unstable focus, surrounded by a unique and repelling cycle.

Proof See Wiggins (1991) and Guckenheimer and Holmes (1983) for a systematic specification of these two curves.

A straightforward economic implication is associated to case (*ii*). That is to say:

Remark 2 For parameter values (τ, β) sufficiently close to $(\bar{\tau}, \bar{\beta})$, the equilibrium associated to the IS-LM model is globally indeterminate. The basin of attraction of the low-growth steady state can be interpreted as a low-growth trapping region.

We can finally conclude that, even though the IS-LM model may exhibit local uniqueness and a determinate saddle path equilibrium, the local analysis is not able to tell the full story, since a deepen investigation reveals that indeterminacy arises in the large, when a global bifurcation analysis is conducted. In particular, when the low-growth steady state is a sink, unless agents anticipate that their destiny will be the high-growth steady state (starting their development close to the high-growth steady state from the beginning), the economy almost always converges (is trapped) to the low-growth steady state (Benhabib *et al*. 2008; Boldrin *et al*. 2001; Mattana *et al*. 2009).

To confirm our results, we will now illustrate some examples and derive the corresponding bifurcation diagrams. The simulations are based on a set of parameter values as in Tu *et al*. (2013) and Makovinyiova (2011), setting $\beta = 0.4$ and leaving τ free to vary. Therefore

Example 1 Consider $\tau = 0.15$. Numerical calculations give $\left(\varepsilon_1, \varepsilon_2\right) = \left(0.05, 0.27\right)$. In this case, the steady state is unique and determinate.

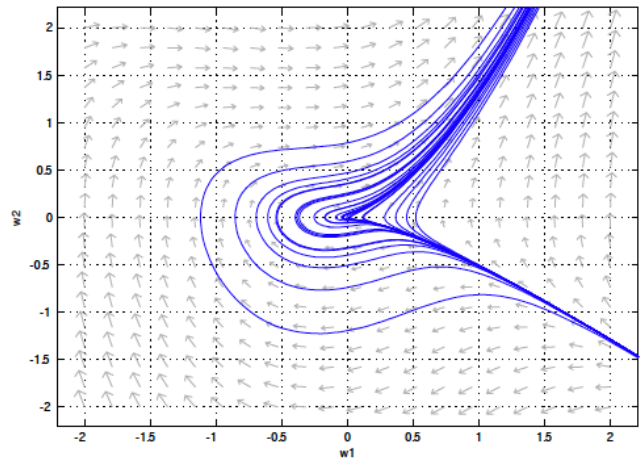

Figure 2: The saddle path steady state.

Example 2 Let now $\tau = 0.22$. Then $\left(\varepsilon_1, \varepsilon_2\right) = \left(-3390, -3149\right)$. In this case, we have two steady states with a heteroclinic connection leading from the non saddle to the saddle equilibrium.

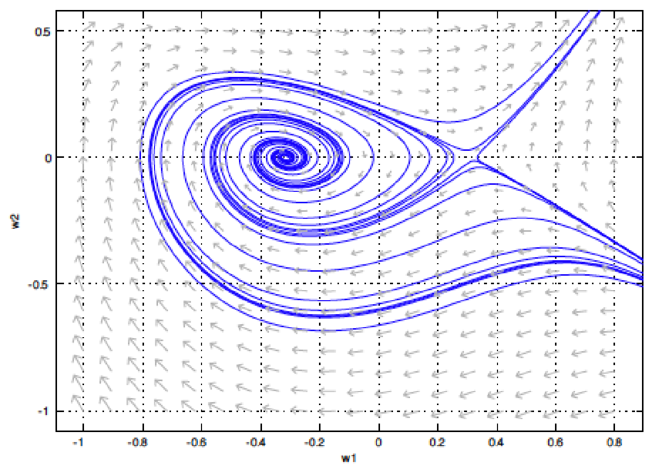

Figure 3: Trapping region with a sink.

Example 3 If alternatively $\tau = 0.32$, it follows that $\left(\varepsilon_1, \varepsilon_2\right) = \left(-5.99, 555.33\right)$. In this case, the model again exhibits two steady states, though the non-saddle steady state is unstable, and surrounded by a sub-critical Hopf cycle.

As clearly depicted in Figures **1-3**, we can conclude that a policy action aimed to lowering the tax rate below its critical value, allows us to stabilize the economy towards a saddle-path stable steady state. Conversely, if we set τ above a certain threshold, i.e. in our model above $\bar{\tau} \simeq 21\%$ of income, an odd situation arises. Namely, indeterminacy occurs, with the economy potentially trapped in a low growth equilibrium with distorsive effects on output due to excessive taxation.

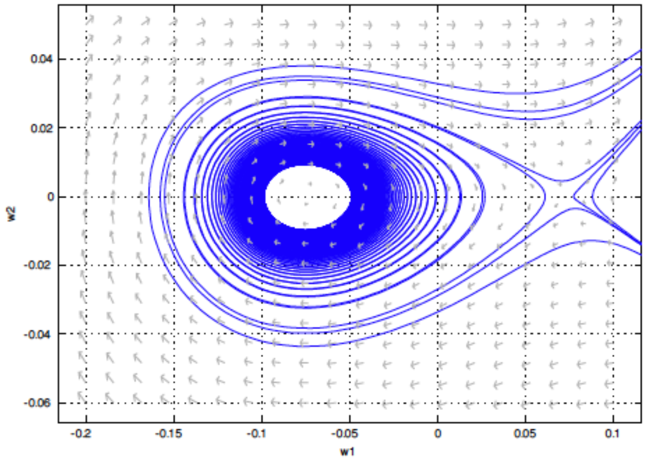

Figure 4: Trapping region with a cycle.

5. CONCLUSIONS

This paper innovates the literature regarding a dynamic IS-LM model of Schinasi's type. First of all, we find that, with a nonlinear function of total inventories, the model admits a dual steady state, characterized by a uniform long-run income level, but different interest rates. One of these steady states is a saddle, and the other is a non-saddle equilibrium.

Once determined the three-dimensional system of differential equations implied by the optimization process, we first provide a characterization of the long-run dynamics. We focus, in particular, on the conditions which allow the emergence of two steady state, one characterized by a relatively high growth rate, and one with a relatively low growth level. Finally, we show that the model undergoes, in plausible regions of the parameters space, a co-dimension 2 Bogdanov-Takens bifurcation. The interesting point is that a generic vector field undergoing a Bogdanov-Takens bifurcation can be put in correspondence of a simple planar system, entirely preserving stability and bifurcation characteristics of the original vector field. The unfolding of this planar system is fully known, and permits the derivation of very useful details regarding the dynamics of any highly nonlinear dynamical system in proximity of the bifurcation. For the scopes of the paper, we are particularly interested in the determination of the regions in the parameters space implying a particular type of global phenomenon, namely the homoclinic bifurcation, by which orbits growing around the non-saddle steady state collide with the saddle steady state. The emergence of this phenomenon is used to establish the possibility of global indeterminacy of the equilibrium for the IS-LM model.

APPENDIX

1. Translation of the Fixed Point to the Origin and Taylor Expansion

Substitute $\tilde{Y} = Y - \bar{Y}^*$, $\tilde{R} = R - \bar{R}^*$, and $\tilde{M} = M - \bar{M}^*$, in system (*P*). Hence

$$\dot{\tilde{R}} = \delta\left[k(\bar{Y}^* + \tilde{Y}) - h(\bar{R}^* + \tilde{R}) - (\bar{M}^* + \tilde{M}) \right]$$

$$\dot{\tilde{Y}} = \alpha \left[\begin{array}{l} \dfrac{(\bar{\beta}+\mu)(\bar{Y}^*+\tilde{Y})^{\eta}}{(\bar{R}^*+\tilde{R})^{\gamma}} - S((\bar{R}^*+\tilde{R}),(1-\bar{\tau}-v) \\ (\bar{Y}^*+\tilde{Y})) + G + \bar{N} - (\bar{\tau}+v+q)(\bar{Y}^*+\tilde{Y}) \end{array} \right] \qquad \text{(A.1)}$$

$$\dot{\tilde{M}} = G + \bar{N} - (\bar{\tau}+v+q)(\bar{Y}^*+\tilde{Y})$$

where $\mu = \beta - \bar{\beta}$, and $v = \tau - \bar{\tau}$, are new trivial variable.

Taylor expanding the system with respect to all 5 variables, we have

$$\begin{pmatrix} \dot{\tilde{R}} \\ \dot{\tilde{Y}} \\ \dot{\tilde{M}} \end{pmatrix} = \mathbf{J} \begin{pmatrix} \tilde{R} \\ \tilde{Y} \\ \dot{\tilde{M}} \end{pmatrix} + \mathbf{B}(\mu, v) \begin{pmatrix} \tilde{R} \\ \tilde{Y} \\ \dot{\tilde{M}} \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{f}_2 \\ 0 \end{pmatrix} \qquad \text{(A.2)}$$

where

$$\mathbf{B}(\mu, v) = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha\gamma \frac{\bar{Y}^{*\,\eta}}{\bar{R}^{*\,\gamma+1}} \mu & \alpha\left[\eta \frac{\bar{Y}^{*\,\eta-1}}{\bar{R}^{*\,\gamma}} \mu - v\right] & 0 \\ 0 & -v & 0 \end{pmatrix} \qquad \text{(A.3)}$$

and

$$\tilde{f}_2 = \frac{\alpha}{2}\left[-\gamma(\gamma+1)\bar{\beta} \frac{\bar{Y}^{*\,\eta}}{\bar{R}^{*\,\gamma+2}} - \bar{S}^*_{RR} \right]$$
$$\tilde{R}^2 + \frac{\alpha}{2}\left[\eta\bar{\beta} \frac{\bar{Y}^{*\,\eta-1}}{\bar{R}^{*\,\gamma}} - (1-\bar{\tau})^2 \bar{S}^*_{YY} \right] \tilde{Y}^2 +$$
$$-\alpha\left[\eta\gamma\bar{\beta} \frac{\bar{Y}^{*\,\eta-1}}{\bar{R}^{*\,\gamma+1}} - (1-\bar{\tau}) \bar{S}^*_{RY} \right] \tilde{R}\tilde{Y}$$

2. Coordinate Change

For double zero eigenvalues, a possible candidate for the eigenbasis is

$$\mathbf{T}=\begin{bmatrix} u_1 & v_1 & z_1 \\ 1 & 0 & z_2 \\ u_3 & v_3 & 1 \end{bmatrix}=\begin{bmatrix} -\frac{j_{22}}{j_{21}} & \frac{1}{j_{21}} & \frac{(j_{11}+j_{33})(j_{11}+j_{22}+j_{33})}{j_{21}j_{32}} \\ 1 & 0 & \frac{j_{11}+j_{22}+j_{33}}{j_{32}} \\ -\frac{j_{12}}{j_{13}}+\frac{j_{11}j_{22}}{j_{21}j_{13}} & -\frac{j_{11}+j_{22}}{j_{21}j_{13}} & 1 \end{bmatrix} \quad \text{(A.4)}$$

where **T** is the transformation matrix. Therefore, operating the coordinate change

$$\begin{pmatrix} \tilde{R} \\ \tilde{Y} \\ \tilde{M} \end{pmatrix}=\mathbf{T}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

the vector field in (A.2) becomes:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{Tr}(\mathrm{J}^*(0)) \end{bmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}+\mathbf{M}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} +\begin{pmatrix} \bar{F}_1\left(w_1,w_2,w_{3,}\right) \\ \bar{F}_2\left(w_1,w_2,w_{3,}\right) \\ \bar{F}_3\left(w_1,w_2,w_{3,}\right) \end{pmatrix} \quad \text{(A.5)}$$

where $\mathbf{M}=\mathbf{T}^{-1}\mathbf{J}\mathbf{T}$, and

$$\bar{F}_i\left(w_1,w_2,w_{3,}\right)=\frac{1}{D}\begin{bmatrix} (-v_1+v_3z_1)Q-v(w_1+z_2w_3)v_1z_2 \\ (u_1-u_3z_1)Q-v(w_1+z_2w_3)(z_1-u_1z_2) \\ (-u_1v_3+v_1u_3)Q+v(w_1+z_2w_3)v_1 \end{bmatrix}$$

with $D=-v_1+v_3z_1-u_1v_3z_2+v_1u_3z_2$ and

$$\begin{aligned} Q=&\frac{\alpha}{2}\left[-\gamma(\gamma+1)\bar{\beta}\frac{\bar{Y}^{*\ \eta}}{\bar{R}^{*\ \gamma+2}}-\bar{S}^*_{RR}\right](u_1w_1+v_1w_2+z_1w_3)^2+ \\ &+\frac{\alpha}{2}\left[\eta\bar{\beta}\frac{\bar{Y}^{*\ \eta-1}}{\bar{R}^{*\ \gamma}}-(1-\bar{\tau})^2\bar{S}^*_{YY}\right](w_1+z_2w_3)^2- \\ &-\alpha\left[\eta\gamma\bar{\beta}\frac{\bar{Y}^{*\ \eta-1}}{\bar{R}^{*\ \gamma+1}}-(1-\bar{\tau})\bar{S}^*_{RY}\right](u_1w_1+v_1w_2+z_1w_3)(w_1+z_2w_3) \end{aligned}$$

3. Computation of the Center Manifold and Normal form for the Non-Linear Part

We now begin the task of reconducting the non-linear part of our vector field to the simplest form. The underlying idea is to perform near-identity transformations to remove the terms that are unessential in the analysis of local dynamical behavior. Recall that after the simplification of the linear part, our vector field, restricted to the center manifold, is the following

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}+\mathbf{A}\left(\mu,\nu\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}+\begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \end{pmatrix} \quad \text{(A.6)}$$

where

$$\mathbf{A}\left(\mu,\nu\right)=\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} \quad \text{(A.7)}$$

with:

$$\xi_1=\tfrac{1}{D}\begin{pmatrix} -\alpha\gamma\frac{\bar{Y}^{*\ \eta}}{\bar{R}^{*\ \gamma+1}}\mu(-v_1+v_3z_1)u_1+ \\ \alpha\left[\eta\frac{\bar{Y}^{*\ \eta-1}}{\bar{R}^{*\ \gamma}}\mu-\nu\right](-v_1+v_3z_1)-v_1z_2\nu \end{pmatrix}$$

$$\xi_2=\tfrac{1}{D}\left(-\alpha\gamma\frac{\bar{Y}^{*\ \eta}}{\bar{R}^{*\ \gamma+1}}\mu(-v_1+v_3z_1)v_1\right)$$

$$\xi_3=\tfrac{1}{D}\begin{pmatrix} -\alpha\gamma\frac{\bar{Y}^{*\ \eta}}{\bar{R}^{*\ \gamma+1}}\mu(u_3z_2-1)u_1+ \\ \alpha\left[\eta\frac{\bar{Y}^{*\ \eta-1}}{\bar{R}^{*\ \gamma}}\mu-\nu\right](u_1-u_3z_1)-(z_1-u_1z_2)\nu \end{pmatrix}$$

$$\xi_4=\tfrac{1}{D}\left(-\alpha\gamma\frac{\bar{Y}^{*\ \eta}}{\bar{R}^{*\ \gamma+1}}\mu(u_3z_2-1)v_1\right)$$

Consider now only the linear and quadratic terms in the variables (w_1,w_2) in our planar vector field

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}+\begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \end{pmatrix} \quad \text{(A.8)}$$

Following the procedure detailed in Freire *et al*. (1989) and Gamero *et al*. (1991), *via* successive transformations, the vector field reduces to the topologically equivalent normal form

$$\begin{cases} \dot{w}_1=w_2+\mathrm{O}(\lvert\cdot\rvert^3) \\ \dot{w}_2=a_2w_1^2+b_2w_1w_2+\mathrm{O}(\lvert\cdot\rvert^3) \end{cases} \quad \text{(A.9)}$$

where

$$a_2=\frac{1}{2}\frac{\partial^2\bar{F}_2}{\partial w_1^2}=\frac{1}{2}\begin{Bmatrix} \alpha\left[-\gamma(\gamma+1)\bar{\beta}\frac{\bar{Y}^{*\ \eta}}{\bar{R}^{*\ \gamma+2}}-\bar{S}^*_{RR}\right]u_1^2+ \\ +\alpha\eta\left[\eta\bar{\beta}\frac{\bar{Y}^{*\ \eta-1}}{\bar{R}^{*\ \gamma}}-(1-\bar{\tau})^2\bar{S}^*_{YY}\right]- \\ 2\alpha\left[\eta\gamma\bar{\beta}\frac{\bar{Y}^{*\ \eta-1}}{\bar{R}^{*\ \gamma+1}}-(1-\bar{\tau})\bar{S}^*_{RY}\right]u_1 \end{Bmatrix}$$

and:

$$\begin{aligned} b_2=\frac{\partial^2\bar{F}_2}{\partial w_1w_2}=&\alpha\left[-\gamma(\gamma+1)\bar{\beta}\frac{\bar{Y}^{*\ \eta}}{\bar{R}^{*\ \gamma+2}}-\bar{S}^*_{RR}\right] \\ &u_1v_1-\alpha\left[\eta\gamma\bar{\beta}\frac{\bar{Y}^{*\ \eta-1}}{\bar{R}^{*\ \gamma+1}}-(1-\bar{\tau})\bar{S}^*_{RY}\right]v_1 \end{aligned}$$

To know how the normal form is affected by the bifurcation parameters β and τ, we need now to find a relationship between the original system and the versal deformation parameters of the reduced system in (A.9). Consider first the matrix $\mathbf{A}(\mu,\nu)$ in (A.7). Easily, it can be shown that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathbf{A}(\mu,\nu)$$

is similar to

$$\mathbf{V}(\mu,\nu) = \begin{pmatrix} 0 & 1 \\ \varepsilon_1 & \varepsilon_2 \end{pmatrix}$$

where

$$\begin{aligned} \varepsilon_1 &= \left(\Lambda_4\mu - \Lambda_5\nu\right)\left(\Lambda_3\mu + 1\right) - \left(\Lambda_1\mu - \Lambda_2\nu\right)\Lambda_6\mu \\ \varepsilon_2 &= (\Lambda_1 + \Lambda_6)\mu - \Lambda_2\nu \end{aligned} \tag{A.10}$$

with

$$\Lambda_1 = \tfrac{1}{D}\left[-\alpha\gamma\tfrac{\bar{Y}^{*\,\eta}}{\bar{R}^{*\,\gamma+1}}(-v_1 + v_3 z_1)u_1 + \alpha\eta\tfrac{\bar{Y}^{*\,\eta-1}}{\bar{R}^{*\,\gamma}}(-v_1 + v_3 z_1)\right]$$

$$\Lambda_2 = \tfrac{1}{D}\left[\alpha(-v_1 + v_3 z_1) + v_1 z_2\right]$$

$$\Lambda_3 = \tfrac{1}{D}\left[-\alpha\gamma\tfrac{\bar{Y}^{*\,\eta}}{\bar{R}^{*\,\gamma+1}}(-v_1 + v_3 z_1)v_1\right]$$

$$\Lambda_4 = \tfrac{1}{D}\left[-\alpha\gamma\tfrac{\bar{Y}^{*\,\eta}}{\bar{R}^{*\,\gamma+1}}(u_3 z_2 - 1)u_1 + \alpha\eta\tfrac{\bar{Y}^{*\,\eta-1}}{\bar{R}^{*\,\gamma}}(u_1 - u_3 z_1)\right]$$

$$\Lambda_5 = \tfrac{1}{D}\left[\alpha(u_1 - u_3 z_1) + (z_1 - u_1 z_2)\right]$$

$$\Lambda_6 = \tfrac{1}{D}\left[-\alpha\gamma\tfrac{\bar{Y}^{*\,\eta}}{\bar{R}^{*\,\gamma+1}}(u_3 z_2 - 1)v_1\right]$$

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